

3. INTEGRATION

UNIT 3.1.1 Basics

- Integration is the reverse of differentiation

1. Since $\frac{d}{dx}(x^2 + c) = 2x$, $\int 2x \, dx = x^2 + c$

2. Since $\frac{d}{dx}(10x + c) = 10$, $\int 10 \, dx = 10x + c$

1. Given $\frac{d}{dx}(x^3 + c) = 3x^2$, find $\int 3x^2 \, dx$. $\int 3x^2 \, dx = \int \left(\frac{d}{dx}(x^3 + c) \right)$ $= x^3 + c$	2. Given $\frac{d}{dx}(2x^5 + c) = 10x^4$, find $\int 10x^4 \, dx$.
3. Given $\frac{d}{dx}\left(\frac{1}{2}x^6 + c\right) = 3x^5$, find $\int 3x^5 \, dx$.	4. Given $\frac{d}{dx}\left(\frac{2}{x} + c\right) = -\frac{2}{x^2}$, find $\int -\frac{2}{x^2} \, dx$.
5. Given $\frac{d}{dx}(4x^3) = f(x)$, find $\int f(x) \, dx$.	6. Given $\frac{d}{dx}(100x + c) = h(x)$, find $\int h(x) \, dx$.
7. Given $\frac{d}{dx}\left(\frac{2}{x-3}\right) = g(x)$, find $\int g(x) \, dx$.	8. Given $\frac{d}{dx}\left(\frac{2x}{x+1}\right) = f(x)$, find $\int f(x) \, dx$.
9. Given $\frac{d}{dx}(x(x+2)^3) = w(x)$, find $\int w(x) \, dx$.	10. Given $\frac{d}{dx}[(x-3)(x+2)] = p(x)$, find $\int p(x) \, dx$.

UNIT 3.1.2 (a) Integration of x^n :

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$$

1. $\int x^3 \, dx = \frac{x^{3+1}}{3+1}$ $= \frac{x^4}{4} + c$	2. $\int x^5 \, dx =$	3. $\int x^9 \, dx =$
4. $\int x^{-3} \, dx =$	5. $\int x^{-2} \, dx =$	6. $\int x \, dx =$

UNIT 3.1.2 (b) Integration of ax^n :

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + c, \quad n \neq -1$$

Note : $\int m dx = mx + c$, m a constant

1. $\int 6x^3 dx = 6 \cdot \frac{x^{3+1}}{3+1} + c$ $= 6 \cdot \frac{x^4}{4} + c$ $= \frac{3x^4}{2} + c$	2. $\int 10x^4 dx =$	3. $\int 4x^3 dx =$
4.. $\int 10 dx = 10x + c$	5. $\int \frac{1}{2} dx =$	6.. $\int -3 dx =$
7. $\int 8x dx = 8 \cdot \frac{x^{1+1}}{1+1} + c$ $= 8 \cdot \frac{x^2}{2} + c$ $= 4x^2 + c$	8. $\int 6x dx =$	9. $\int 3x dx =$
10. $\int 12x^3 dx =$	11. $\int 8x^2 dx =$	12. $\int 10x^5 dx =$
13. $\int \frac{2}{x^3} dx = \int 2x^{-3} dx$ $= 2 \cdot \frac{x^{-3+1}}{-3+1} + c$ $= 2 \cdot \frac{x^{-2}}{-2} + c$ $= -\frac{1}{x^2} + c$	14. $\int \frac{8}{x^5} dx = \int 8x^{-5} dx$ $=$	15. $\int \frac{12}{x^4} dx =$
16. $\int \frac{2}{5x^3} dx =$	17. $\int \frac{2}{3} x dx =$	18. $\int 0.9x^2 dx =$

UNIT 3.1.3 To Determine Integrals of Algebraic Expressions.

Note : Integrate term by term. Expand & simplify the given expression where necessary.

Example :
$$\int (3x^2 - 4x + 5)dx = \frac{3x^3}{3} - \frac{4x^2}{2} + 5x + c$$

$$= x^3 - 2x^2 + 5x + c$$

1. $\int (6x - 4)dx$ =	2. $\int (12x^2 + 8x - 1)dx$ =	3. $\int (x^3 - 3x + 2)dx$ =
4. $\int x(3x - 2)dx$ =	5. $\int (2x - 1)(2x+1)dx$ =	6. $\int (x+2)(x-3)dx$ =
7. $\int (3x - 2)^2 dx$ =	8. $\int \frac{(2x - 1)(2x+1)}{x^2} dx$ =	9. $\int \frac{6x^2 - 4}{x^2} dx$ =
10. $\int \frac{(3x + 4)^2}{x^2} dx$ =	11. $\int (2x^{-2} - x + 1)dx$ =	12. $\int (2 - x)^2 dx$ =
13. $\int (3 - \frac{1}{x})(3 + \frac{1}{x})dx$ =	14. $\int (-2x^{-2} + 3 - x + \frac{1}{x^2})dx$ =	15. $\int x(3 - x)^2 dx$ =

UNIT 3.1.4 To Determine the Constant of Integration (I)

Example		Exercise	
1	<p><i>SPM 2003, K2, S3(a) 3 Marks</i></p> <p>Given $\frac{dy}{dx} = 2x + 2$ and $y = 6$ when $x = -1$, find y in terms of x.</p> <p>Answer: $\frac{dy}{dx} = 2x + 2$</p> $y = \int (2x + 2) dx$ $= \frac{2x^2}{2} + 2x + c$ $y = x^2 + 2x + c$ <p>$y = 6, x = 1,$ $6 = 1^2 + 2(1) + c$</p> $6 = 3 + c$ $c = 3$ <p>Hence $y = x^2 + 2x + 3$</p>	2	<p>Given $\frac{dy}{dx} = 2x + 3$ and $y = 4$ when $x = 1$, find y in terms of x.</p> <p style="text-align: right;">$y = x^2 + 3x$</p>
3	<p>Given $\frac{dy}{dx} = 4x + 1$ and $y = 4$ when $x = -1$, find y in terms of x.</p> <p>Answer : $\frac{dy}{dx} = 4x + 1$</p> $y = \int (4x + 1) dx$ $=$ <p style="text-align: right;">$y = 2x^2 + x + 3$</p>	4	<p>Given $\frac{dy}{dx} = 6x - 3$ and $y = 3$ when $x = 2$, find y in terms of x.</p> <p style="text-align: right;">$y = 3x^2 - 3x - 3$</p>
5	<p>Given $\frac{dy}{dx} = 4 - 2x$ and $y = 5$ when $x = 1$, find y in terms of x.</p> <p style="text-align: right;">$y = 4x - x^2 + 2$</p>	6	<p>Given $\frac{dy}{dx} = 3x^2 - 2$ and $y = 4$ when $x = -1$, find y in terms of x.</p> <p style="text-align: right;">$y = x^3 - 2x + 3$</p>
7	<p>Given $\frac{dy}{dx} = 3(2x - 4)$ and $y = 30$ when $x = -2$, find y in terms of x.</p> <p style="text-align: right;">$y = 3x^2 - 12x - 6$</p>	8	<p>Given $\frac{dy}{dx} = 2 - \frac{3}{x^2}$ and $y = 1$ when $x = -1$, find y in terms of x.</p> <p style="text-align: right;">$y = 2x + \frac{3}{x} + 6$</p>

UNIT 3.1.4 To Determine the Constant of Integration (II)

Example		Exercise	
1	<p>Given $\frac{dy}{dx} = 6x - 4$ and $y = 2$ when $x = -1$, find the value of y when $x = 2$.</p> <p>Answer: $\frac{dy}{dx} = 6x - 4$</p> $y = \int (6x - 4) dx$ $= \frac{6x^2}{2} - 4x + c$ $y = 3x^2 - 4x + c$ <p>$y = 2, x = -1,$ $2 = 3(-1)^2 - 4(-1) + c$</p> $2 = 3 + 4 + c$ $c = -5$ $y = 3x^2 - 4x - 5$ <p>When $x = 2,$ $y = 3(2)^2 - 4(2) - 5 = -1$</p>	2	<p>Given $\frac{dy}{dx} = 2x + 5$ and $y = 1$ when $x = -1$, find the value of y when $x = 3$.</p>
		29	
3	<p>Given $\frac{dy}{dx} = 1 - 3x^2$ and $y = 2$ when $x = -1$, find the value of y when $x = 2$.</p> <p>Answer: $\frac{dy}{dx} = 1 - 3x^2$</p> $y = \int (1 - 3x^2) dx$ $=$	4	<p>Given $\frac{dy}{dx} = 6x - 3$ and $y = 3$ when $x = 2$, find the value of y when $x = 1$.</p>
		-4	-3
5	<p>Given $\frac{dy}{dx} = 4 - 2x$ and $y = 5$ when $x = 1$ find the value of y when $x = -1$.</p>	6	<p>Given $\frac{dy}{dx} = 3x^2 - 2$ and $y = 4$ when $x = -1$, find the value of y when $x = 0$.</p>
		-3	3
7	<p>Given $\frac{dy}{dx} = 3(2x - 1)$ and $y = 5$ when $x = -2$, find the value of y when $x = 1$.</p>	8	<p>Given $\frac{dy}{dx} = 5 - \frac{3}{x^2}$ and $y = 1$ when $x = -1$, find the value of y when $x = 3$</p>
		-13	25

3.1.5 To Determine the Equation of Curve from Gradient Function

Example		Exercise	
1	<p><i>SPM 2004, K2, S5 (3 Marks)</i> The gradient function of a curve which passes through A(1, -12) is $3x^2 - 6x$. Find the equation of the curve.</p> <p><i>Answer:</i></p> $\frac{dy}{dx} = 3x^2 - 6x$ $y = \int (3x^2 - 6x) dx$ $= \frac{3x^3}{3} - \frac{6x^2}{2} + c$ $y = x^3 - 3x^2 + c$ <p>$y = -12, x = 1,$ $-12 = 1^3 - 3(1) + c$</p> $-12 = -2 + c$ $c = -10$ <p>Hence $y = x^3 - 6x - 10$</p>	2.	<p>The gradient function of a curve which passes through B(1, 5) is $3x^2 + 2$. Find the equation of the curve.</p> <p><i>Answer:</i></p> $\frac{dy}{dx} = 3x^2 + 2$ $y = \int (3x^2 + 2) dx$ $=$ $y = x^3 + 2x + 2$
3.	<p>The gradient function of a curve which passes through P(1, -3) is $4x - 6$. Find the equation of the curve.</p> <p><i>Answer:</i></p> $y = 2x^2 - 6x + 1$	4.	<p>The gradient function of a curve which passes through Q(-1, 4) is $3x(x - 2)$. Find the equation of the curve.</p> <p><i>Answer:</i></p> $y = x^3 - 3x^2 + 8$
5.	<p>The gradient function of a curve which passes through A(1, 6) is $5 - 3x^2$. Find the equation of the curve.</p> $y = 5x - x^3 + 2$	6.	<p>The gradient function of a curve which passes through R(2, 3) is $6x^2 - 4$. Find the equation of the curve.</p> $y = x^3 - 4x + 3$
7.	<p>The gradient function of a curve which passes through A(1, 10) is $x(6 - 3x)$. Find the equation of the curve.</p> $y = 3x^2 - x^3 + 8$	8.	<p>The gradient function of a curve which passes through A(-1, 7) is $3x^2 + 2x - 1$. Find the equation of the curve.</p> $y = x^3 + x^2 - x + 6$

UNIT 3.1.6 Integration of the form $\int (ax+b)^n dx, n \neq -1$

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c, n \neq -1$$

E X A M P L E	$\int (3x+2)^4 dx = \frac{(3x+2)^{4+1}}{3(4+1)} + c$ $= \frac{(3x+2)^5}{15} + c$	E X A M P L E	$\int \frac{12}{(2x-3)^4} dx = \int 12(2x-3)^{-4} dx$ $= \frac{12(2x-3)^{-3}}{-3 \cdot (2)} + c$ $=$
1.	$\int (2x+5)^3 dx =$	2.	$\int \frac{4}{(x-3)^2} dx =$
3.	$\int (2+4x)^5 dx =$	4.	$\int (x+2)^{-4} dx =$
5.	$\int \frac{3}{(2x-1)^2} dx =$	6.	$\int \frac{15}{(3x-5)^6} dx =$
7.	$\int 6(2-x)^3 dx =$	8.	$\int 30(4+3x)^{-3} dx =$
9.	$\int \frac{2}{3}(1-2x)^3 dx =$	10.	$\int \frac{15}{2(x-3)^4} dx =$

UNIT 3.2 Definite Integral

If $\int f(x)dx = F(x) + c$, then $\int_a^b f(x)dx = F(b) - F(a)$

<p>1. $\int_1^2 2x \, dx = [x^2]_1^2$ $= 2^2 - 1^2$ $= 4 - 1$ $= 3$</p>	<p>2. $\int_0^3 4x^3 \, dx = [x^4]_0^3$ $=$</p>	<p>3. $\int_1^2 6x^2 \, dx =$ $=$</p>
[81]		[14]
<p>4. $\int_1^2 8x \, dx =$ $=$</p>	<p>5. $\int_2^4 x^3 \, dx =$ $=$</p>	<p>6. $\int_1^3 10 \, dx =$ $=$</p>
[12]	[60]	[20]
<p>7. $\int_1^2 3x^{-2} \, dx =$ $=$</p>	<p>8. $\int_1^3 \left(\frac{2}{x^3}\right) \, dx =$</p>	<p>9. $\int_1^2 \frac{3}{2x^2} \, dx =$ $=$</p>
[$\frac{3}{2}$]	[$\frac{8}{9}$]	[$\frac{3}{4}$]
<p>10. $\int_0^3 (2 - 6x) \, dx =$</p>	<p>11. $\int_1^3 (4x - 3x^2) \, dx =$</p>	<p>12. $\int_0^3 x(2x + 1) \, dx =$</p>
[-21]	[-10]	[22.5]
<p>13. $\int_1^2 (2x - 1)(2x + 1) \, dx$ $=$</p>	<p>14. $\int_1^3 (3x - 2)^2 \, dx$ $=$</p>	<p>15. $\int_0^1 x(3x - 2) \, dx =$</p>
[$\frac{25}{3}$]	[38]	[0]

3.2.1 Definite Integral of the form $\int (ax + b)^n dx, n \neq -1$

<p style="text-align: center;">E X A M P L E</p>	$\int_0^1 (3x+2)^4 dx = \left[\frac{(3x+2)^5}{5 \cdot 3} \right]_0^1$ $= \left[\frac{(3x+2)^5}{15} \right]_0^1$ $= \frac{5^5}{15} - \frac{2^5}{15}$ $= 206.2$	<p style="text-align: center;">Y o u T r y !</p>	$\int_1^2 \frac{3}{(2x-1)^2} dx = \int_1^2 3(2x-1)^{-2} dx$ $=$ $=$
<p>1.</p>	$\int_0^1 (2x+3)^2 dx =$ <p style="text-align: right;">[16 $\frac{1}{3}$]</p>	<p>2.</p>	$\int_4^5 \frac{10}{(x-3)^2} dx =$ <p style="text-align: right;">[5]</p>
<p>3.</p>	$\int_0^1 16(2+4x)^3 dx =$ <p style="text-align: right;">[1280]</p>	<p>4.</p>	$\int_0^1 6(x+2)^{-3} dx =$ <p style="text-align: right;">[$\frac{5}{12}$]</p>
<p>5.</p>	$\int_1^2 \frac{6}{(2x-1)^2} dx =$ <p style="text-align: right;">[2]</p>	<p>6.</p>	$\int_2^3 \frac{24}{(3x-5)^3} dx =$ <p style="text-align: right;">[3.75]</p>

UNIT 3.2.3 Applications of Definite Integral (I)

Given that $\int_1^3 f(x)dx = 4$, $\int_1^3 g(x)dx = 6$. Find the value of	
1. $\int_3^1 f(x)dx =$	2. $\int_1^3 [2 + f(x)]dx =$
-4	8
3. $\int_1^3 [5f(x)]dx =$	4. $\int_1^3 [f(x) - 4]dx =$
20	-4
5. $\int_1^3 [4f(x) - 2x]dx =$	6. $\int_1^3 \left[\frac{3f(x) - 1}{2} \right] dx =$
8	5
7. $\int_1^3 [3f(x) - 2g(x)]dx =$	8. $\int_1^3 \left[2g(x) + \frac{1}{2}f(x) \right] dx =$
0	14
9. $\int_1^3 \frac{[6f(x) - g(x)]}{3} dx =$	10. $\int_1^3 \left[\frac{3}{2}f(x) + 2g(x) - 1 \right] dx =$
6	16

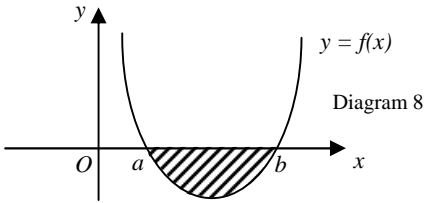
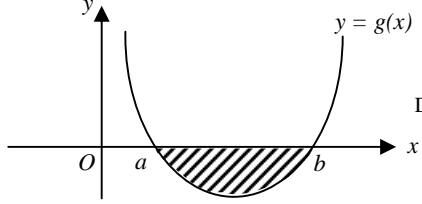
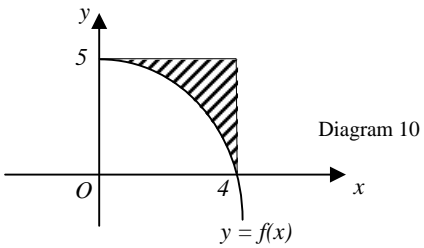
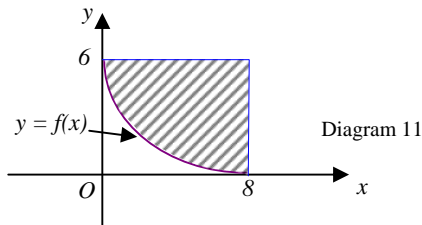
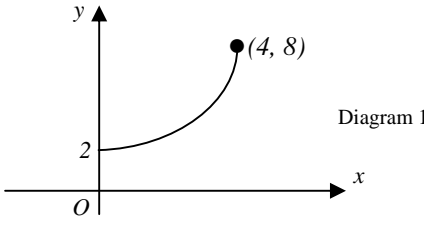
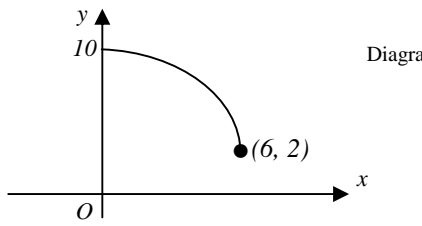
UNIT 3.2.3 Applications of Definite Integral (II)

<p>1. Given that $\int_{-1}^k (2x-3)dx = 6$, where $k > -1$, find the value of k. (SPM 2004, P1, Q 22)</p> <p style="text-align: right;">$k = 4$</p>	<p>2. Given that $\int_0^k (2x-1)dx = 12$, where $k > 0$, find the value of k.</p> <p style="text-align: right;">4</p>
<p>3. Given that $\int_0^k (3-4x)dx = -20$, where $k > 0$, find the value of k.</p> <p style="text-align: right;">4</p>	<p>4. Given that $\int_0^k (6x+1)dx = 14$, where $k > 0$, find the value of k.</p> <p style="text-align: right;">2</p>
<p>5. Given that $\int_2^6 f(x) = 7$ and $\int_2^6 (2f(x) - kx) = 10$, find the value of k. [SPM '05, P1, Q21]</p> <p style="text-align: right;">$\frac{1}{4}$</p>	<p>6. Given that $\int_1^3 f(x) = 5$ and $\int_1^3 (3f(x) + kx) = 23$, find the value of k.</p> <p style="text-align: right;">2</p>
<p>5. Given that $\int_2^5 f(x) = 6$ and $\int_2^5 (f(x) + kx) = 10$, find the value of k.</p> <p style="text-align: right;">$\frac{8}{21}$</p>	<p>6. Given that $\int_1^4 f(x) = 3$ and $\int_1^4 (kf(x) + 6x) = 39$, find the value of k. [k=-2]</p> <p style="text-align: right;">-2</p>

Questions based on SPM Format (I)

1.	Find the value of $\int_1^2 \frac{3-x^2}{x^2} dx$.	2	Find the value of $\int_0^3 x(4-x) dx$.
	1/2		9
3.	Find the value of $\int_0^2 \frac{5dx}{(1+2x)^2}$.	4	Find the value of $\int_1^4 \frac{15}{(3x-2)^2} dx$.
	2		4.5
5.	Given that $\int_1^k (2-2x)dx = -16$, where $k > 0$, find the value of k.	6.	Given that $\int_0^k (4x-1)dx = 6$, where $k > 0$, find the value of k.
	$k = 5$		$k = 2$
7.	Given that $\int_0^4 f(x) = 15$ and $\int_0^4 (2f(x) - kx) = 6$, find the value of k.	8.	Given that $\int_1^4 f(x) = 10$ and $\int_1^4 (3f(x) - kx) = 0$, find the value of k.
	$k = 3$		$k = 4$

SPM FORMAT QUESTIONS (III)

<p>1. (SPM 06, P1, Q20) Diagram 8 shows the curve $y = f(x)$ cutting the x-axis at $x = a$ and $x = b$.</p>  <p>Diagram 8</p> <p>Given that the area of the shaded region is 5 unit^2, find the value of $\int_a^b 2f(x)dx$.</p> <p style="text-align: right;">Answer :</p>	<p>2. Diagram 9 shows the curve $y = f(x)$ cutting the x-axis at $x = a$ and $x = b$.</p>  <p>Diagram 9</p> <p>Given that the area of the shaded region is 6 unit^2, find the value of $\int_b^a 3g(x)dx$.</p> <p style="text-align: right;">Answer :</p>
<p>3. Diagram 10 shows part of the curve $y = f(x)$.</p>  <p>Diagram 10</p> <p>Given that $\int_0^4 f(x)dx = 15 \text{ unit}^2$, find the area of the shaded region.</p> <p style="text-align: right;">Answer :</p>	<p>4. Diagram 11 shows part of the curve $y = f(x)$.</p>  <p>Diagram 11</p> <p>Given that the area of the shaded region is 40 unit^2, find the value of $\int_0^8 f(x)dx$.</p> <p style="text-align: right;">Answer :</p>
<p>5. (SPM 01) Diagram 12 shows the sketch of part of a curve.</p>  <p>Diagram 12</p> <p>(a) Shade, on the given diagram, the region represented by $\int_2^8 x dy$.</p> <p>(b) Find the value of $\int_0^4 y dx + \int_2^8 x dy$</p> <p style="text-align: right;">Answer : (b)</p>	<p>6. Diagram 13 shows the sketch of part of a curve.</p>  <p>Diagram 13</p> <p>(a) Shade, on the given diagram, the region represented by $\int_2^{10} x dy$.</p> <p>(b) If $\int_2^{10} x dy = p$, find, in terms of p, the value of $\int_0^6 y dx$.</p> <p style="text-align: right;">Answer : (b)</p>

SPM FORMAT QUESTIONS (IV)

<p>1.</p>	<p>(SPM'03) Given that $\int \frac{5}{(1+x)^4} dx = k(1+x)^n + c$, find the value of k and n. [3 marks]</p> <p style="text-align: right;">$[k = -\frac{5}{3}, n = -3]$</p>	<p>2.</p>	<p>Given that $\int \frac{12}{(3x-2)^3} dx = k(3x-2)^n + c$, find the value of k and n.</p> <p style="text-align: right;">$k = -2, n = -2$</p>
<p>3.</p>	<p>(SPM 02) Given that $4x + 3\frac{dy}{dx} = 6$, express y in terms of x.</p> <p style="text-align: right;">$y = 2x - \frac{2}{3}x^2 + c$</p>	<p>4.</p>	<p>Given that $8x - 2\frac{dy}{dx} = 5$, express y in terms of x.</p> <p style="text-align: right;">$y = 2x^2 - \frac{5}{2}x + c$</p>
<p>5.</p>	<p>(SPM 1999) Given that $\frac{d^2y}{dx^2} = 5x^4 + 1$, when $x = 1, y = -3$ and $\frac{dy}{dx} = -2$, find y in terms of x.</p> <p style="text-align: right;">$y = \frac{1}{6}x^6 + \frac{1}{2}x^2 - 4x + \frac{1}{3}$</p>	<p>6.</p>	<p>Given that $\frac{d^2y}{dx^2} = 2 - 6x$, when $x = -1, y = 3$ and $\frac{dy}{dx} = -1$, find y in terms of x.</p> <p style="text-align: right;">$y = x^2 - x^3 + 4x + 5$</p>

7.	<p>(SPM 2002) Given that $\int_0^1 (16x^2 + 10kx + k^2) dx = \frac{4}{3}$. Find the possible values of k.</p> <p style="text-align: right;">$k = -1, -4$</p>	8.	<p>Given that $\int_0^1 (3x^2 + 10kx + 4) dx = 0$. Find the value of k.</p> <p style="text-align: right;">$k = -1$</p>
9.	<p>(SPM 01) Given that $\frac{d}{dx} \left(\frac{x^2}{x-1} \right) = g(x)$, find the value of $\int_2^3 (2x - g(x)) dx$.</p> <p style="text-align: right;">$\frac{9}{2}$</p>	10	<p>Given that $\frac{d}{dx} \left(\frac{x}{x-1} \right) = f(x)$, find the value of $\int_2^3 (4x + f(x)) dx$.</p> <p style="text-align: right;">$13\frac{1}{2}$</p>
11.	<p>Given that $\frac{d}{dx} \left(\frac{x-1}{x+2} \right) = h(x)$, find the value of $\int_0^1 (6 - 4h(x)) dx$.</p> <p style="text-align: right;">[4]</p>	12	<p>Given that $\frac{d}{dx} \left(\frac{12}{(x+1)^3} \right) = w(x)$, find the value of $\int_0^1 (6w(x) - 3) dx$.</p> <p style="text-align: right;">[- 66]</p>